Grassmannians

Big picture

→ have worked hard at general theory of affine & proj. varieties

→ now: take a break and look at cool class of examples

Have seen: $\mathbb{P}^n = \{ \ell \subseteq K^{n+1} : \ell \text{ is a } 1\text{-dim'l linear subspace} \}$

gave structure of variety to this set

⇝ moduli space

Natural generalization: look at k-dim'l linear subspaces

Def (Grassmannians)

Let $n \in \mathbb{N}_{>0}$, $0 \leq k \leq n$.

$$\mathrm{Gr}(k,n) = \{ L \subseteq K^n : L \text{ is a } k\text{-dim'l linear subspace} \}$$

⇝ <u>Grassmannian</u> of k-planes in K^n.

Rmk · $\mathrm{Gr}(1,n) = \mathbb{P}^{n-1}$

· Under $K^n \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^{n-1}$: $L \cap (K^n \setminus \{0\}) \longrightarrow \pi(L) = \mathbb{P}(L)$

k-dim'l linear space

$(k-1)$ dim'l projective linear subspace

⇝ can interpret $G(k,n)$ as set of $(k-1)$-dim'l proj. lin. subspaces in $\mathbb{P}^{n-1}$

Goal Give $\mathrm{Gr}(k,n)$ structure of variety

Strategy Write down injective map

$$\varphi : \mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}^N$$

large number N

and show that image of φ is closed ⇝ $\mathrm{Gr}(k,n)$ = proj. variety.

Need Coordinates on $Gr(K,n)$, unique / scaling.

Outlook

For $Gr(1,n) = \mathbb{P}^n$ we chose generator $L = Lin(v)$ K-linear span
$\rightsquigarrow$ components of v unique / scaling

Similar: For $Gr(k,n)$ choose basis $L = Lin(v_1, v_2, \dots, v_k)$

$\rightsquigarrow$ get matrix
$$M = \begin{array}{c} \begin{pmatrix} v_{1,1} & v_{2,1} & \cdots & v_{k,1} \\ v_{1,2} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ v_{1,n} & v_{2,n} & \cdots & v_{k,n} \end{pmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{Columns} = \text{basis of } L \end{array} \in K^{n\times k}$$

Unique / colum operations
= right-multiplic. by $S \in GL(k,K)$

For $I \subseteq \{1,\dots,n\}$ a collection of k rows
$\rightsquigarrow$ minor p_I = determinant of $k\times k$-submatrix M_I
$\rightsquigarrow$ vector
$$P_L = (p_I)_{\substack{I\subseteq\{1,\dots,n\} \\ |I|=k}} \in K^{\binom{n}{k}}$$

$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \rightsquigarrow$ $p_{\{1,2\}} = 1\cdot4 - 2\cdot3 = -2$
$p_{\{1,3\}} = 1\cdot6 - 2\cdot5 = -4$
$p_{\{2,3\}} = 3\cdot6 - 4\cdot5 = -2$

Replacing $M \to M\cdot S$ $\rightsquigarrow$ $\det((M\cdot S)_I) = \det(M_I \cdot S)$
$= \det(M_I)\cdot\det(S)$
$\rightsquigarrow$ all p_I scaled by $\det(S)$

$\Rightarrow [P_L] \in \mathbb{P}^{\binom{n}{k}-1}$ is well-defined ♀

Map $Gr(k,n) \longrightarrow \mathbb{P}^{\binom{n}{k}-1}$
$L \longmapsto [P_L]$

Plücker embedding
injective, closed image ✓

Note: Discussion above uses coordinates, matrices, determin., ...
$\rightsquigarrow$ give cleaner description using alternating tensor products

Alternating tensor products

Def Let V, W be K-vector spaces, $k \in \mathbb{N}$.
$f: V^k \longrightarrow W$ (k-fold) multilinear map
f is called alternating if $f(v_1, \dots, v_k) = 0$ for all $v_1, \dots, v_k \in V$ s.t. $v_i = v_j$ for some $i \neq j$.

Rmk $f: V^k \to W$ alternating multilinear

$$\Rightarrow \underset{\text{alternat.}}{0} = f(\dots, v_i + v_j, \dots, v_i + v_j, \dots)$$

$$\overset{\text{multilinear}}{=} \underbrace{f(\dots, v_i, \dots, v_i, \dots)}_{=0} + f(\dots, v_i, \dots, v_j, \dots) + f(\dots, v_j, \dots, v_i, \dots) + \underbrace{f(\dots, v_j, \dots, v_j, \dots)}_{=0}$$

$$\Rightarrow f(\dots, v_i, \dots, v_j, \dots) = - f(\dots, v_j, \dots, v_i, \dots)$$

⇝ exchanging two arguments $\hat{=}$ multiplying by (-1).

$\Rightarrow$ For any permutation $\sigma \in S_k$:

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sign}(\sigma) \cdot f(v_1, \dots, v_k).$$

Exa (a) $\det : \mathrm{Mat}(n \times n, K) = (K^n)^n \longrightarrow K$
is alternating n-fold multilin. map to $W = K$.

(b) The cross-product

$$f: K^3 \times K^3 \longrightarrow K^3$$

$$\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$ ⇝ basically Plücker coordinates!

is an alternating bilinear map.

Def A k-fold alternating tensor product of V is K-vect. space T with k-fold altern. map $\tau: V^k \to T$ such that:

$$\begin{tikzcd} V^k \arrow[r, "\forall f"] \arrow[d, "\tau"'] & W \\ T \arrow[ur, dashed, "\exists! g"'] & \end{tikzcd}$$

any K-vector space

$$\left(\begin{array}{l} \forall f: V^k \to W \ k\text{-fold alternating} \\ \exists \text{ unique } g: T \to W \text{ linear}: f = g \circ \tau \end{array} \right)$$ ← universal property.

Pro (Existence & uniqueness of alt. tensor products)
For any vector space V and $k \in \mathbb{N}$ there is a k-fold alternating tensor product $\tau : V^k \to T$ and it is unique up to unique isomorphism.

Notation
$T =: \Lambda^k V$, $\tau(v_1, \dots, v_k) =: v_1 \wedge \dots \wedge v_k \in \Lambda^k V$ $(v_i \in V)$.

Pf sketch Can take $T = \underbrace{V \otimes \dots \otimes V}_{k \text{ times}} / L$
with
$L = \mathrm{Span}_K \{ v_1 \otimes \dots \otimes v_k : \exists i \neq j : v_i = v_j \}$

$$\begin{tikzcd} T & V^{\otimes k} \arrow[l] \\ & V \times \dots \times V \arrow[u] \arrow[ul, "\tau"] \end{tikzcd}$$ $\square$

Exa V K-vector space of dim. $n \in \mathbb{N}$, $e_1, \dots, e_n$ basis of V
(a) $v_1, \dots, v_k \in V \rightsquigarrow v_1 \wedge \dots \wedge (v_i + v_j) \wedge \dots \wedge (v_i + v_j) \wedge \dots \wedge v_k = 0 \in \Lambda^k V$
same comput. as above $\rightsquigarrow$ $v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)} = \mathrm{sgn}(\sigma) \cdot v_1 \wedge \dots \wedge v_k$ for $\sigma \in S_k$.

(b) Basis of $V \otimes \dots \otimes V = V^{\otimes k}$:
$e_{i_1} \otimes \dots \otimes e_{i_k}$ for $1 \leq i_1, \dots, i_k \leq n$ $\quad$ $\dim V^{\otimes k} = n^k$

Basis of $\Lambda^k V$:
$e_{i_1} \wedge \dots \wedge e_{i_k}$ for $1 \leq i_1 < i_2 < \dots < i_k \leq n$ $\quad$ $\dim \Lambda^k V = \binom{n}{k}$

(c) **Exa** $V = K^3$, $v = a_1 e_1 + a_2 e_2 + a_3 e_3$, $w = b_1 e_1 + b_2 e_2 + b_3 e_3$
$\rightsquigarrow v \wedge w = a_1 b_1 \cdot \underbrace{e_1 \wedge e_1}_{=0} + a_1 b_2 \cdot e_1 \wedge e_2 + \dots + a_2 b_1 \cdot \underbrace{e_2 \wedge e_1}_{= -e_1 \wedge e_2} + \dots$
$= (a_1 b_2 - a_2 b_1) \cdot e_1 \wedge e_2 + (a_1 b_3 - a_3 b_1) e_1 \wedge e_3 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3$.

(d) $\Lambda^0 V \cong K$, $\Lambda^1 V \cong V$, $\Lambda^n V$ has dim 1 by (b)
$\rightsquigarrow$ isomorphism $\Lambda^n V \xrightarrow[\det]{\sim} K$

Determinants & linear dependence

Aim Coordinates describing lin. spaces $L \in Gr(k,n)$

$\rightsquigarrow$ see that we can use $\Lambda^k K^n$!

Rmk (Alternating tensor products and determinants)

$0 \leq k \leq n$, $v_1, \dots, v_k \in V = K^n$ with $v_i = \sum_{j=1}^{n} a_{ij} e_j$.

Q For $v_1 \wedge \dots \wedge v_k$, what is coeff. of $e_{i_1} \wedge \dots \wedge e_{i_k}$ $(1 \leq i_1 < \dots < i_k \leq n)$?

A Can calculate

$$v_1 \wedge \dots \wedge v_k \overset{\text{multilin.}}{=} \sum_{j_1 \dots j_k} a_{1 j_1} \cdots a_{k j_k} \cdot e_{j_1} \wedge \dots \wedge e_{j_k}$$

group terms with same set of e_{i_j} and reorder.

$$= \sum_{1 \leq i_1 < \dots < i_k \leq n} \underbrace{\left(\sum_{\sigma \in S_k} a_{1, i_{\sigma(1)}} \cdots a_{k, i_{\sigma(k)}} \cdot \mathrm{sgn}(\sigma) \right)}_{\text{determinant of sub-matrix of } A \text{ (rows } i_1, \dots, i_k)} e_{i_1} \wedge \dots \wedge e_{i_k}$$

$n \times k$ matrix

$$A = \begin{array}{cc} \left(\begin{array}{ccc} a_{11} & \cdots & a_{k1} \\ a_{12} & \cdots & a_{k2} \\ \boxed{\quad} & & \\ \vdots & & \vdots \\ \boxed{\quad} & & \\ a_{1n} & \cdots & a_{kn} \end{array} \right) & \begin{array}{c} \\ \\ \leftarrow i_1 \\ \vdots \\ \leftarrow i_k \\ \\ \end{array} \end{array}$$

columns $\hat{=}$ $v_1, \dots, v_k$

$\Rightarrow$ coeff. of $e_{i_1} \wedge \dots \wedge e_{i_k}$ = maximal minors of matrix A.

$\hookrightarrow$ c.f. "Outlook" in first lecture

Lem $v_1, \dots, v_k \in K^n$, then

$v_1 \wedge \dots \wedge v_k = 0 \in \Lambda^k K^n \iff v_1, \dots, v_k$ linearly dependant in K^n.

Pf $v_1 \wedge \dots \wedge v_k = 0 \overset{\text{Rmk}}{\iff}$ all maximal minors of A vanish

$\overset{\text{LinAlg.}}{\iff}$ A does not have full rank $\iff$ v_i lin. dep. $\square$

Consequence $v_1 \wedge \dots \wedge v_k \neq 0 \rightsquigarrow L = \mathrm{Lin}(v_1, \dots, v_k) \in Gr(k,n)$

Next Show that $v_1 \wedge \dots \wedge v_k$ uniquely characterizes L! (up to scaling)

Lem $v_1,\dots,v_k \in K^n$ and $w_1,\dots,w_k \in K^n$ sets of lin. indep. vectors.
Then

$v_1\wedge\dots\wedge v_k$ and $w_1\wedge\dots\wedge w_k$ linearly dependent in $\Lambda^k V$
$\Longleftrightarrow \operatorname{Lin}(v_1,\dots,v_k) = \operatorname{Lin}(w_1,\dots,w_k) \in Gr(k,n)$.

Pf $\{v_i\}_i$ and $\{w_j\}_j$ lin. indep. $\xRightarrow{\text{Lem}}$ $v_1\wedge\dots\wedge v_k,\ w_1\wedge\dots\wedge w_k \neq 0$

"$\Rightarrow$" Assume $v_1\wedge\dots\wedge v_k = \lambda\cdot w_1\wedge\dots\wedge w_k$ for $\lambda\in K$

$\overset{w_i\wedge -}{\rightsquigarrow}$ $w_i\wedge v_1\wedge\dots\wedge v_k = \lambda\cdot w_i\wedge w_1\wedge\dots\wedge w_k \overset{w_i \text{ appears twice}}{=} 0 \quad \forall i$

$\xRightarrow{\text{Lem}}$ w_i lin. dependent of $v_1,\dots,v_k$ $\forall i$ $\Rightarrow \operatorname{Lin}(w_1,\dots,w_k)\subseteq \operatorname{Lin}(v_1,\dots,v_k)$
$\rightsquigarrow$ "=" for dim. reasons.

"$\Leftarrow$" $\operatorname{Lin}(v_1,\dots,v_k) = \operatorname{Lin}(w_1,\dots,w_k)$
$\rightsquigarrow \exists B\in GL(k,K)$: $w_i = \sum b_{ij} v_j$

$$\Rightarrow w_1\wedge\dots\wedge w_k = \sum_{j_1,\dots,j_k} b_{1j_1}\cdots b_{kj_k}\cdot v_{j_1}\wedge\dots\wedge v_{j_k} \overset{\text{as above}}{=\!=} \underbrace{\det(B)}_{\neq 0}\cdot v_1\wedge\dots\wedge v_k$$

$\square$

The Plücker embedding

Can use $\Lambda^k K^n$ to see $Gr(k,n)$ as subset of proj. space.

Construction (Plücker embedding)
Let $0 \le k \le n$.

$$\rightsquigarrow f: Gr(k,n) \longrightarrow \mathbb{P}^{\binom{n}{k}-1}$$
$$\mathrm{Lin}(v_1,\dots,v_k) \longmapsto [v_1\wedge\dots\wedge v_k]$$

$v_1\wedge\dots\wedge v_k \in \Lambda^k K^n \cong K^{\binom{n}{k}}$

Note:

- well-defined : $v_1\wedge\dots\wedge v_k \neq 0$ as v_i lin. indep.
  $\mathrm{Lin}(v_1,\dots,v_k) = \mathrm{Lin}(w_1,\dots,w_k) \overset{\text{Lem}}{\Longrightarrow} v_1\wedge\dots\wedge v_k$ and $w_1\wedge\dots\wedge w_k$ lin. dependent.
- injective : $\overset{\text{Lem}}{\Longleftarrow}$ above.

Notation f **Plücker embedding**

homog. coordinates of $f(L)$: **plücker coordinates**

maximal minors of $A = \begin{pmatrix} v_1 \cdots v_k \end{pmatrix}$

$\rightsquigarrow$ consider $Gr(k,n) \subseteq \mathbb{P}^{\binom{n}{k}-1}$ via f.

Exa

(a) For $Gr(1,n)$:

$$f(\mathrm{Lin}(a_1e_1+\dots+a_ne_n)) = [\underbrace{a_1e_1+\dots+a_ne_n}_{\in \Lambda^1 K^n \cong K^n}] = (a_1:\dots:a_n) \in \mathbb{P}^{\binom{n}{1}-1} = \mathbb{P}^{n-1}.$$

$\Rightarrow Gr(1,n) = \mathbb{P}^{n-1}$ as expected.

(b) $L = \mathrm{Lin}(e_1+e_2, e_1+e_3) \in Gr(2,3)$

$(e_1+e_2)\wedge(e_1+e_3) = -e_1\wedge e_2 + e_1\wedge e_3 + e_2\wedge e_3$

$\rightsquigarrow f(L) = (-1:1:1) \in \mathbb{P}^{\binom{3}{2}-1} = \mathbb{P}^2$

maximal minors.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

Change of basis of L

$\hat{=}$ column operations on $A = A_L$

$\rightsquigarrow$ minors change by constant factor.

Grassmannians as projective varieties

Have seen $Gr(k,n) \subseteq \mathbb{P}^{\binom{n}{k}-1}$ via Plücker embedding

Now: show $Gr(k,n)$ closed subset $\rightsquigarrow$ projective variety.

Note

$$Gr(k,n) = \{ [\underbrace{v_1 \wedge \dots \wedge v_k}_{\text{pure tensor}}] \in \mathbb{P}^{\binom{n}{k}-1} : v_1, \dots, v_k \text{ lin. indep.} \}$$

(general elem. of $\Lambda^k K^n$: linear combin. of these)

$\rightsquigarrow$ want: equations cutting out set of pure tensors in $\Lambda^k K^n$.

Lem For $0 \neq \omega \in \Lambda^k K^n$ with $k < n$ consider

$$f : K^n \longrightarrow \Lambda^{k+1} K^n, \quad v \mapsto v \wedge \omega \qquad \text{K-linear map}$$

Then $\operatorname{rk} f \geq n-k$ with equality iff. $\omega = v_1 \wedge \dots \wedge v_k$ for some $v_1, \dots, v_k \in K^n$.

Exa $k=2$, $n=4$

(a) $\omega = e_1 \wedge e_2$

$$\rightsquigarrow f(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) = (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge e_1 \wedge e_2$$
$$= a_3 \cdot e_1 \wedge e_2 \wedge e_3 + a_4 \cdot e_1 \wedge e_2 \wedge e_4$$

$\rightsquigarrow \operatorname{rk}(f) = 2 = 4-2$

(b) $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$

$$f(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) = (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)$$
$$= a_3 \cdot e_1 \wedge e_2 \wedge e_3 + a_4 \cdot e_1 \wedge e_2 \wedge e_4$$
$$+ a_1 \cdot e_1 \wedge e_3 \wedge e_4 + a_2 \cdot e_2 \wedge e_3 \wedge e_4$$

$\rightsquigarrow \operatorname{rk}(f) = 4 > 4-2 = 2$

$\rightsquigarrow \omega$ is not a pure tensor

Pf of Lem Want: $f: K^n \longrightarrow \Lambda^{k+1} K^n$, $v \mapsto v \wedge \omega$

$\Rightarrow$ rk $f \geq n-k$ with equality iff. $\omega = v_1 \wedge \dots \wedge v_k$ for some $v_1, \dots, v_k \in K^n$.

Idea Compute rk(f) on suitable basis

Let $v_1, \dots, v_r$ be a basis of ker(f), $r = n - \text{rk}(f)$

$\rightsquigarrow$ extend to basis $v_1, \dots, v_n$ of K^n

$\rightsquigarrow$ Can write

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \underbrace{a_{i_1 \dots i_k}}_{\in K} \cdot \underbrace{v_{i_1} \wedge \dots \wedge v_{i_k}}_{\to \text{ basis of } \Lambda^k K^n}$$

$$\overset{v_i \in \ker(f)}{\underset{i=1,\dots,r}{\Longrightarrow}} \quad 0 = v_i \wedge \omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1 \dots i_k} \cdot \underbrace{v_i \wedge v_{i_1} \wedge \dots \wedge v_{i_k}}_{}, \quad i = 1, \dots, r$$

$= 0$ if $i \in \{i_1, \dots, i_k\}$; $\to$ basis elem. of $\Lambda^{k+1} K^n$ otherwise (up to sign)

$\Rightarrow a_{i_1 \dots i_k} = 0$ if $i \notin \{i_1, \dots, i_k\}$

Holds for all $i = 1, \dots, r \rightsquigarrow a_{i_1 \dots i_k} = 0$ unless $\{1 \dots r\} \subseteq \{i_1 \dots i_k\}$

$\omega \neq 0 \Rightarrow r \leq k \iff \text{rk}(f) = n - r \geq n - k$.

Equality $\Rightarrow$ only possible nonzero coeff. $= a_{1 \dots r}$

$\Rightarrow \omega = (a_{1 \dots r} v_1) \wedge v_2 \wedge \dots \wedge v_k$ pure tensor

Conversely: $\omega = w_1 \wedge \dots \wedge w_k$ for w_i lin. indep.

$\rightsquigarrow w_1, \dots, w_k \in \ker(f) \Rightarrow \dim \ker(f) \geq k \Rightarrow \text{rk}(f) \leq n-k$

other ineq. $\geq$ above. $\square$

Cor (Gr(k,n) as a projective variety)

Gr$(k,n) \subseteq \mathbb{P}^{\binom{n}{k}-1}$ is a closed subset, hence a proj. variety.

Pf $k = n \rightsquigarrow \text{Gr}(n,n) = \{[K^n]\}$ closed $\rightsquigarrow$ assume $k < n$.

$$\text{Gr}(k,n) = \{[\omega] \in \mathbb{P}^{\binom{n}{k}-1} : \omega \text{ pure tensor}\}$$

$$\overset{\text{Lem}}{=} \{[\omega] \in \mathbb{P}^{\binom{n}{k}-1} : \text{rank}(K^n \xrightarrow{f} \Lambda^{k+1} K^n, v \mapsto v \wedge \omega) \overset{\leq}{=} n-k\}$$

enough!

homogen. polynomials in coord. $a_{i_1 \dots i_k}$ of ω.

$\hookrightarrow$ vanishing = closed condition

$\Big\{$ all $(n-k+1) \times (n-k+1)$ minors of matrix(f) vanish. $\square$

Exa $k=2,\ n=4 \rightsquigarrow \varphi: K^4 \longrightarrow \Lambda^3 K^4 \cong K^4$

$$v \longmapsto v \wedge \overbrace{\left(a_{12}\, e_1\wedge e_2 + a_{13}\, e_1 \wedge e_3 + \cdots + a_{34}\, e_3 \wedge e_4\right)}^{\omega \in \Lambda^2 K^4}$$

$$e_1 \longmapsto a_{23} \cdot e_1\wedge e_2\wedge e_3 + a_{24}\, e_1\wedge e_2\wedge e_4 + a_{34}\, e_1\wedge e_3\wedge e_4$$

$$\rightsquigarrow \mathrm{Mat}(\varphi) = \begin{array}{cc} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} & \\ \begin{pmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{pmatrix} & \begin{array}{l} e_1\wedge e_2\wedge e_3 \\ e_1\wedge e_2\wedge e_4 \\ e_1\wedge e_3\wedge e_4 \\ e_2\wedge e_3\wedge e_4 \end{array} \end{array}$$

16 3×3-minors = hom. degree 3 in a_{ij}

$\rightsquigarrow$ cut out $Gr(2,4)$ in $\mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$

[Exercise 8.22(a)] $Gr(2,4)$ cut out by single quadratic equation!

Affine charts & further properties of Grassmannians

Know $Gr(k,n) \subseteq \mathbb{P}^{\binom{n}{k}-1}$ and $\mathbb{P}^{\binom{n}{k}-1} = V_0 \cup \dots \cup V_{\binom{n}{k}-1}$

⇝ gives affine cover $U_i = V_i \cap Gr(k,n)$

In fact: can understand these patches explicitly ⇝ top. properties of $Gr(k,n)$

Construction

Let $U_0 \subseteq Gr(k,n)$ be open subset where $e_1 \wedge \dots \wedge e_k$- coord. nonzero

⇝ $L \in Gr(k,n)$ in $U_0 \iff L$ is column-span of matrix

$$\begin{pmatrix} A \\ \hdashline B \end{pmatrix}$$

A → $k \times k$ invertible matrix ($\det = e_1 \wedge \dots \wedge e_k$-coeff $\neq 0$)

B → $(n-k) \times k$ matrix

$\cdot A^{-1}$ from right

column-operat. don't change L

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \hdashline & BA^{-1} & \end{pmatrix}$$

$\Rightarrow U_0$ is image of the map

$$f: \mathbb{A}^{(n-k)\times k} \longrightarrow U_0 \subseteq Gr(k,n)$$

$$\overset{\text{Mat}(n-k,k)}{=} \quad C \longmapsto \text{column span of } \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \hdashline & C & \end{pmatrix}.$$

- Argument above $\Rightarrow$ $\boxed{f \text{ surjective}}$
- Plücker coord. of $f(C)$ = max. minors of $\begin{pmatrix} 1 \cdots 1 \\ \hdashline C \end{pmatrix}$ = polynomials in entries of C

  $\Rightarrow$ $\boxed{f \text{ morphism}}$

- Can reconstruct c_{ij} as minor of $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \hdashline & C & \end{pmatrix}$, up to ± 1

take rows $(1, \dots, \hat{i}, \dots, k, j+k)$

omit ith row

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \rightsquigarrow \begin{array}{l} c_{12} = \det\begin{pmatrix} 1 & 0 \\ c_{11} & c_{12} \end{pmatrix} \\ c_{11} = -\det\begin{pmatrix} 0 & 1 \\ c_{11} & c_{12} \end{pmatrix} \end{array}$$

$\Rightarrow$ $\boxed{f \text{ injective, } f^{-1} \text{ morphism}}$

$\Rightarrow f$ isomorphism ⇝ $\mathbb{A}^{(n-k)\cdot k} \overset{\text{open}}{\subseteq} Gr(k,n)$

Other patches U_i: same argument

⇝ $Gr(k,n)$ has cover by spaces $\mathbb{A}^{(n-k)\times k}$

Cor $Gr(k,n)$ is an irreducible projective variety of dimension $(n-k)\cdot k$.

Pf $Gr(k,n)$ covered by open sets $\mathbb{A}^{(n-k)\cdot k}$ ← irred. of dim. $(n-k)\cdot k$
any two have non-trivial intersection
find $n\times k$-matrix such that all $k\times k$-minors are nonzero
$\Rightarrow$ result from [Exerc. 2.21(b), 2.34(a)]. □

Rmk $U = \{A \in Mat(n,k) : rk(A) = k\} \overset{\text{open}}{\subseteq} Mat(n,k) = \mathbb{A}^{n\cdot k}$

$$\begin{array}{c} U \\ g \downarrow \\ Gr(k,n) \end{array} \qquad A \mapsto \text{column-span of } A$$

$\rightsquigarrow$ g is morphism, invariant under $GL(k,K) \overset{\text{right-multiplication}}{\curvearrowright} U$

In suitable sense: $Gr(k,n) = U/GL(k,K)$.

Symmetry property:

Pro For $0 \le k \le n$ we have $Gr(k,n) \cong Gr(n-k,n)$.

Pf Have bijection
$f: Gr(k,n) \longrightarrow Gr(n-k,n)$, → $f^{-1}(\tilde{L}) = \tilde{L}^\perp$
$L \mapsto L^\perp = \{x \in K^n : \langle x,y\rangle = 0 \ \forall y \in L\}$

$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$ ← bilinear form

Remains: f is a morphism (similar: f^{-1})
Check in affine coordinates: $L \in Gr(k,n)$ column-span of $\left(\begin{array}{c} E_k \\ \hline C \end{array}\right)$
Have:
$$\left(-C \,\vdots\, E_{n-k}\right)\cdot\left(\begin{array}{c} E_k \\ \hline C \end{array}\right) = 0 \Rightarrow L^\perp = \text{row space of } \left(-C \,\vdots\, E_{n-k}\right)$$

Plücker coord. of $L^\perp$ = max minors of $\left(-C \,\vdots\, E_{n-k}\right)$
= polyn. in entries of C $\Rightarrow$ f morphism.
□